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GEOMETRICAL DESCRIPTION OF THE LOCAL INTEGRALS OF MOTION OF MAXWELL-BLOCH EQUATION

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We represent a classical Maxwell-Bloch equation and related to it positive part of the AKNS hierarchy in geometrical terms. The Maxwell-Bloch evolution is given by an infinitesimal action of a nilpotent subalgebra $\mathbf{n}_+$ of affine Lie algebra $\widehat{sl}_2$ on a Maxwell-Bloch phase space treated as a homogeneous space of $\mathbf{n}_+$. A space of local integrals of motion is described using cohomology methods. We show that hamiltonian flows associated to the Maxwell-Bloch local integrals of motion (i.e. positive AKNS flows) are identified with an infinitesimal action of an abelian subalgebra of the nilpotent subalgebra $\mathbf{n}_+$ on a Maxwell- Bloch phase space. Possibilities of quantization and latticization of Maxwell-Bloch equation are discussed.

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1. Introduction

A powerful cohomological technique was proposed by B. Feigin and E. Frenkel [1,2] for description of a space of local integrals of motion for Toda field theories associated to affine Lie algebra $\mathbf{g}$ [3,4]. It was proved that the phase space of the classical Toda theories was isomorphic to the quotient N_+/A_+, where N_+ was the Lie group of a nilpotent Lie subalgebra $\mathbf{n}_+$ of algebra $\mathbf{g}$, and A_+ is its principle commutative subgroup. This identification gave possibility to interprete the evolution of the Toda system on its phase space in simple terms. Namely, Toda Hamiltonian action on the phase space coincided with the left infinitesimal action on the nilpotent subalgebra $\mathbf{n}_+$ on N_+/A_+ , and action of Toda local integrals of motion was given by the right infinitesimal action of a commutative subalgebra $\mathbf{a}_-$, opposite to the principal commutative subalgera $\mathbf{a}_+$ with respect to the Cartan involution. The space of local integrals of motion of Toda theory (quantum and classical) was ungerstood as a cohomology class of certain complex. We transfer this technique to the simpliest case of classical Maxwell-Bloch (MB) equation [3,5,6].

Here we will use the following simple form of classical MB equation

$$\partial_\tau \beta = e^{-\int^z dt\,\beta(t)\gamma(t)},$$
$$\partial_\tau \gamma = e^{\int^z dt\,\beta(t)\gamma(t)},$$

where $\beta(z)$, $\gamma(z)$ are functions on the circle $|z| = 1$, $z \in \mathbf{C}$. MB eq. can be rewritten in the zero-curvature form (compare to [5])

$$B_1 = \lambda\sigma_3 + \begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix},$$
$$C_1 = -\frac{1}{4}\lambda^{-1} e^{-\int^z dt\,\beta(t)\gamma(t)} \begin{pmatrix} \gamma & -\gamma^2 \\ 1 & -\gamma \end{pmatrix},$$
$$\partial_\tau B_1 - \partial C_1 - [B_1, C_1] = 0,$$

where $\partial = \partial/\partial z$, σ_3 is Pauli matrix and $r,\ q$ are functions $r = \frac{1}{2}\beta,\ q = \frac{1}{2}(-\beta\gamma^2 + 2\partial\gamma)$.

MB eq. belongs to the AKNS hierarchy [3]

$$B_n(\lambda) = \sum_{j=0}^{n} b_j \lambda^{n-j},\ \ C_n(\lambda) = \sum_{j=0}^{n-1} c_j \lambda^{j-n},\ \ n = 1, 2, \cdots$$

where $b_0 = \sigma_3$; b_j , c_j are matrixes 2×2.

$$\partial_{x_n} B_m - \partial_{x_m} B_n + [B_m, B_n] = 0,$$
$$\partial_{y_n} C_m - \partial_{y_m} C_n + [C_m, C_n] = 0,$$
$$\partial_{y_n} B_m - \partial_{x_m} C_n + [B_m, C_n] = 0,$$

for $m, n = 1, 2, \cdots$. In MB eq. $\partial_{x_1} \equiv \partial,\ \partial_{y_1} \equiv \partial_\tau$. We call the first line of this system as positive (with respect to the spectral parameter λ) part of the AKNS hierarchy.

MB eq. is the Hamilton equation

$$\partial_\tau \beta = \{\beta,\, H\}, \quad \partial_\tau \gamma = \{\gamma,\, H\}$$

with hamiltonian

$$\oint dz \beta(z) e^{\int^z dt\, \beta(t)\gamma(t)} - \oint dz \gamma(z) e^{-\int^z dt\, \beta(t)\gamma(t)}$$

and Poisson brackets

$$\begin{aligned}
\{\gamma(x), \beta(y)\} &= \delta(x-y) + \frac{1}{2}\epsilon(x-y)\gamma(x)\beta(y), \\
\{\gamma(x), \gamma(y)\} &= -\frac{1}{2}\epsilon(x-y)\gamma(x)\gamma(y), \\
\{\beta(x), \beta(y)\} &= -\frac{1}{2}\epsilon(x-y)\beta(x)\beta(y),
\end{aligned}$$

where ϵ is sign function. In this paper we investigate local integrals of motion of MB eq., i.e. local functionals of the form $\int dz P(\beta(z), \gamma(z))$, where P is a differential polinomial in β, γ. It is known that local integrals of motion of MB eq. coincide with the hamiltonians of the positive part of the AKNS hierarchy.

Consider affine algebra $\widehat{sl}_2$ without central extension as defined over formal Laurent power series $\mathbf{C}((t))$: $\widehat{sl}_2 = sl_2 \otimes \mathbf{C}((t)) \oplus \mathbf{C}d$, d is standart $\mathbf{Z}$-grading operator, $d = td/dt$. Let e, h, f be standart sl_2 generators; then define Cartan subalgebra $\mathbf{h}$ of the affine algebra $\widehat{sl}_2$: $\mathbf{h} = \mathbf{C}h \otimes 1 \oplus \mathbf{C}d$. Nilpotent subalgebras $\mathbf{n}_+ (\mathbf{n}_-)$ of $\widehat{sl}_2$ are generated by $e, f \otimes t$ (resp. $f, e \otimes t^{-1}$). For $\widehat{sl}_2$ we have Cartan decomposition $\widehat{sl}_2 = \mathbf{n}_+ \oplus \mathbf{h} \oplus \mathbf{n}_-$. Denote $\mathbf{h}_\pm$ as commuting subalgebra of $\mathbf{n}_\pm$, generated by $h_{\pm i} = h \otimes t^{\pm i}, \quad i = 1, 2, \cdots$. Let N_+ be the Lie group of the nilpotent subalgebra $\mathbf{n}_+$ and H_+ be the Lie group of the commutative subalgebra $\mathbf{h}_+$.

We will show that the MB phase space is isomorphic to the quotient N_+/H_+.

The action of two parts of MB hamiltonian on the phase space is identified with the left infinitesimal action of the nilpotent subalgebra $\mathbf{n}_+$ on the quotient N_+/H_+. Then we interpret the space of local integrals of motion of MB eq. as the first cohomology of $\mathbf{n}_+$ with coefficients in $\mathbf{C}(N_+/H_+)$. This formulation allows to describe the space of the MB local integrals of motion and gives opportunity to quantize them [1]. Moreover the action of local integrals of motion (i.e. positive AKNS hamiltonians) on the phase space can be realized as the right infinitesimal action of the commutative subalgebra $\mathbf{h}_-$ on the quotient N_+/H_+.

This paper is organized as follows. In Section 2 we introduce the MB phase space as reduced bosonic β, γ, ϕ system. The MB hamiltonian is written. We show that an action of the hamiltonian gives rise to an action of nilpotent subalgebra

$\mathbf{n}_+$ of $\widehat{sl}_2$ on the phase space. In Section 3 geometrical picture of MB eq. is given. We prove that there exists the $\mathbf{n}_+$-isomorphism between MB phase space and quoutient N_+/H_+. In Section 4 we give cohomological description of the space of local integrals of motion. In Section 5 we show that vector fields generated by the local integrals of motion of MB eq. are given by the right infinitesimal action of $\mathbf{h}_-$ on N_+/H_+.

2. Definition and Hamiltonian Picture

In this section we give the MB eq. as a reduction of the β, γ, ϕ system. Consider three classical bosonic fields $\beta(z), \gamma(z), \partial\phi(z)$ on circle $z \in \mathbf{C}$, $|z| = 1$ with Poisson brackets

$$\begin{aligned} \{\gamma(z), \beta(w)\} &= \delta(z-w), \\ \{\phi(z), \phi(w)\} &= \frac{1}{2}\epsilon(z-w). \end{aligned} \tag{1}$$

Let Π_0 be space of differential polinomials in $\beta, \gamma, \partial\phi$, i.e. polinomials of the form

$$P(\beta(z), \gamma(z), \partial\phi(z), \partial\beta(z), \partial\gamma(z), \partial^2\phi(z), \cdots),$$

$(\partial = \partial/\partial z)$ We will refer to

$$F[\beta, \gamma, \phi] = \oint dz\, P\left(\beta(z), \gamma(z), \partial\phi(z), \partial\beta(z), \partial\gamma(z), \partial^2\phi(z), \cdots\right),$$

as local functionals. Denote space of local functionals as $\widehat{\Pi}_0$. The space $\widehat{\Pi}_0$ can be identified with $\Pi_0/\partial\Pi_0$, because integral of total derivative is zero. We can ever consider space of functions of the form $P(\beta, \gamma, \partial\phi, \cdots)e^{n\phi}$, where $P \in \Pi_0$, which can be treated as $\Pi_n = \Pi_0 \otimes e^{n\phi}$. Denote $\partial' = \partial + n\partial\phi$. It is obvious that $\partial(Pe^{n\phi}) = (\partial' P)e^{n\phi}$, so the action of ∂ on Π_n is $(\partial + n\partial\phi) \otimes 1$. Define $\widehat{\Pi}_n$ as integrals of elements from Π_n , i.e. $\widehat{\Pi}_n = \Pi_n/\partial\Pi_n$.

Introduce a hamiltonian

$$H = \oint dz\, \beta(z)e^{\phi(z)} \; - \; \oint dz\, \gamma(z)e^{-\phi(z)} \in \widehat{\Pi}_1 \oplus \widehat{\Pi}_{-1} \tag{2}$$

Consider the Hamilton equation with Poisson brackets (1) and hamiltonian (2)

$$\begin{aligned} \partial_\tau \beta &= \{\beta,\, H\} = e^{-\phi}, \\ \partial_\tau \gamma &= \{\gamma,\, H\} = e^{\phi}, \\ \partial_\tau \partial\phi &= \{\partial\phi,\, H\} = \beta e^{\phi} + \gamma e^{-\phi}, \end{aligned} \tag{3}$$

where ∂_τ designates the derivative over the time τ.

Define the action of operators $\bar{Q}_1, \bar{Q}_0$ on the space Π_0

$$
\begin{aligned}
\bar{Q}_1 : \Pi_0 \longrightarrow \Pi_1 \ , \ \ \bar{Q}_1 &= \left\{ \cdots, \oint dz\, \beta e^{\phi} \right\}, \\
\bar{Q}_0 : \Pi_0 \longrightarrow \Pi_{-1} \ , \ \ \bar{Q}_0 &= \left\{ \cdots, \oint dz\, \gamma e^{-\phi} \right\}.
\end{aligned}
\tag{4}
$$

It is easy to see that operators $\bar{Q}_1, \bar{Q}_0$ commute with the action of ∂. So the action of these operators is well- defined on $\widehat{\Pi}_0$.

Comparing definition (4) with hamiltonian (2) , we see that evolution (3) is given by the operator

$$
\partial_\tau = \bar{Q}_1 - \bar{Q}_0 \tag{5}
$$

A local functional $I \in \widehat{\Pi}_0$ is called a local integral of motion of the system (3), if $\partial_\tau I = 0$.The local integrals of motion will be designated as IM. We see from definition (4) that the space of IM of system (3) can be written as:

$$
\widehat{\Pi}_0 \supset \text{the space of IM of system (3)} = \ \text{Ker } \bar{Q}_1 \cap \ \text{Ker } \bar{Q}_0,
$$

because the operator ∂_τ maps an element from $\widehat{\Pi}_0$ to an element from the sum of two different spaces $\widehat{\Pi}_1 \oplus \widehat{\Pi}_{-1}$.

Introducing translator operator T

$$
T : \ \Pi_n \longrightarrow \Pi_{n+1} \, , \ P \otimes e^{n\phi} \longrightarrow P \otimes e^{(n+1)\phi},
$$

where $P \in \Pi_0$, define vector fields Q_1, Q_0 on Π_0

$$
\begin{aligned}
Q_1 &= \bar{Q}_1 T^{-1} \\
Q_0 &= \bar{Q}_0 T
\end{aligned}
\tag{6}
$$

Denote the expression $\beta\gamma - \partial\phi$ as J. The evolution (3) of the function J is trivial

$$
\partial_\tau J(z) = 0
$$

Thus we can reduce the space Π_0, imposing the condition

$$
J(z) = 0, \tag{7}
$$

which is compatible with evolution (3). Reduction (7) of the system (3) gives Maxwell-Bloch equation

$$
\begin{aligned}
\partial_\tau \beta &= e^{-\int^z dt\, \beta(t)\gamma(t)}, \\
\partial_\tau \gamma &= e^{\int^z dt\, \beta(t)\gamma(t)}.
\end{aligned}
\tag{8}
$$

Now we describe Poisson structure of $\beta, \gamma, \partial\phi$ system reduced by the condition $J = 0$. Let ue define

$$\pi_0 = \Pi_0 / \text{ D Pol}(J)\Pi_0$$

where D Pol is a set of differential polinomials.

Namely, a differential polinomial $P(\beta, \gamma, \partial\phi)$ from the space π_0 is equivalent to

$$P(\beta, \gamma, \partial\phi) \sim P(\beta, \gamma, \partial\phi) + \sum_{n=0}^{\infty} R_n \, \partial^n J, \tag{9}$$

where R_n is differential polinomial in $\beta, \gamma, \partial\phi$. Introduce a Dirac bracket on π_0 $\{\,,\,\}^*$ with the main property

$$\{P(\beta, \gamma, \partial\phi), J\}^* = 0,$$

for any differential polinomial P. Then we have: if A, B are differential polinomials in $\beta, \gamma, \partial\phi$, such that $A \sim B$, then $\{\xi, A\}^* \sim \{\xi, B\}^*$ for arbitrary ξ. Using this property and equivalence formula (9) we can treat π_0 as Poisson manifold with coordinates $\beta, \gamma, \partial\beta, \partial\gamma$ etc and Poisson-Dirac bracket:

$$\begin{aligned}
\{\gamma(x), \beta(y)\}^* &= \delta(x - y) + \frac{1}{2}\epsilon(x - y)\gamma(x)\beta(y), \\
\{\gamma(x), \gamma(y)\}^* &= -\frac{1}{2}\epsilon(x - y)\gamma(x)\gamma(y), \\
\{\beta(x), \beta(y)\}^* &= -\frac{1}{2}\epsilon(x - y)\beta(x)\beta(y),
\end{aligned} \tag{10}$$

where ϵ is sign function.

Poisson structure (10) is the first Poisson structure $\{\ ,\ \}_1$ for NLS eq. [7]. After reduction (7) space Π_n transforms to π_n

$$\pi_n = \pi_0 \otimes \exp(n \int^z d\tau \, \beta\gamma).$$

The action of derivative ∂ on π_n is following

$$(\partial + n\beta\gamma) \otimes 1 \tag{11}$$

Define $\hat{\pi}_n = \pi_n / \partial\pi_n$ as a space of functionals of the form

$$\oint dz \ \text{ D Pol}(\beta, \gamma) e^{n \int^z dt \, \beta\gamma}.$$

After reduction (7) hamiltonian (2) transforms as follows

$$\oint dz \, \beta(z) e^{\int^z \ dt \, \beta(t)\gamma(t)} - \oint dz \, \gamma(z) e^{-\int^z \ dt \, \beta(t)\gamma(t)} \ \in \hat{\pi}_1 \otimes \hat{\pi}_{-1}. \tag{12}$$

The action of operators $\bar{Q}_1$ and $\bar{Q}_0$ is compatible with reduction (7) , i.e. $\bar{Q}_1 J = \bar{Q}_0 J = 0$. Thus they can be limited from Π_0 to π_0

$$\begin{aligned}
\bar{Q}_1 : \pi_0 \longrightarrow \pi_1 \ , \ \ \bar{Q}_1 &= \left\{ \cdots , \oint dz\, \beta e^{\int^z \, dt\, \beta(t)\gamma(t)} \right\}^* , \\
\bar{Q}_0 : \pi_0 \longrightarrow \pi_{-1} \ , \ \ \bar{Q}_0 &= \left\{ \cdots , \oint dz\, \gamma e^{-\int^z \, dt\, \beta(t)\gamma(t)} \right\}^* .
\end{aligned} \tag{13}$$

The explicit formula for the action (13) of operators $\bar{Q}_1$ and $\bar{Q}_0$ on the element $P \in \pi_0$ is

$$\begin{aligned}
\bar{Q}_1 P &= \sum_{n\geq 0} B_n^+ \frac{\partial P}{\partial(\partial^n \gamma)} \otimes e^{\int^z \, dt\, \beta(t)\gamma(t)}, \\
\bar{Q}_0 P &= \sum_{n\geq 0} B_n^- \frac{\partial P}{\partial(\partial^n \beta)} \otimes e^{-\int^z \, dt\, \beta(t)\gamma(t)},
\end{aligned} \tag{14}$$

where $P, B_n^\pm \in \pi_0$ and

$$\begin{aligned}
\partial_z^n e^{\int^z \, dt\, \beta(t)\gamma(t)} &= B_n^+ e^{\int^z \, dt\, \beta(t)\gamma(t)}, \\
\partial_z^n e^{-\int^z \, dt\, \beta(t)\gamma(t)} &= B_n^- e^{-\int^z \, dt\, \beta(t)\gamma(t)}.
\end{aligned}$$

The MB eq. (8) is treated as Hamilton equation

$$\begin{aligned}
\partial_\tau \beta &= \{\beta, \, H\}^*, \\
\partial_\tau \gamma &= \{\gamma, \, H\}^*
\end{aligned}$$

with brackets $\{\ ,\ \}^*$ (10) and hamiltonian (12).

A local functional $I \in \widehat{\pi}_0$ is called a local integral of motion of MB eq., if $\partial_\tau I = 0$. After imposing the reduction (7) on the formula for evolution (5) we get that the space of IM of MB eq. *(8) is the intersection of kernels of operators $\bar{Q}_1$, $\bar{Q}_0$

$$\hat{\pi}_0 \supset \ \text{the space of IM of MB eq.} = \ \text{Ker}\ \bar{Q}_1 \cap \ \text{Ker}\ \bar{Q}_0$$

Using translator operator T

$$T : \ \pi_n \longrightarrow \pi_{n+1} \ , \ \ P \otimes e^{n \int^z dt\, \beta\gamma} \longrightarrow P \otimes e^{(n+1)\int^z dt\, \beta\gamma},$$

where $P \in \pi_0$, and explicit formula (14) define vector fields Q_1, Q_0 on π_0

$$\begin{aligned}
Q_1 &= \bar{Q}_1 T^{-1} = \sum_{n\geq 0} B_n^+ \frac{\partial}{\partial(\partial^n \gamma)}, \\
Q_0 &= \bar{Q}_0 T = \sum_{n\geq 0} B_n^- \frac{\partial}{\partial(\partial^n \beta)}.
\end{aligned} \tag{15}$$

Vector fields Q_1 and Q_0 on π_0 satisfy Serre relations for the nilpotent subalgebra $\mathbf{n}_+$

$$ad^3_{Q_0} \cdot Q_1 = 0 \text{ and } ad^3_{Q_1} \cdot Q_0 = 0$$

and can be identified with the generators of the nilpotent subalgebra $\mathbf{n}_+$: e and $f \otimes t$.

Thus vector fields Q_1 and Q_0 give structure of $\widehat{sl}_2$ module to π_0.

Define grading on π_0:

$$\begin{aligned}
&deg\beta = (s=-1, q=1), \;\; deg\gamma = (s=0, q=-1), \\
&deg\partial = (s=-1, q=0), \;\; deg v_n = (s = -\frac{1}{2}n(n-1), q=n),
\end{aligned} \tag{16}$$

where q is the isospin with respect to current J: $\{J(x), P_q(y)\} = q\delta(x-y)P_q(y)$

Then $deg\bar{Q}^*_1 = deg\bar{Q}^*_0 = 0$.

3. Geometrical Picture

In this section we give geometrical description of the space π_0. Recall that we consider affine algebra $\widehat{sl}_2$ without central extension as defined over formal Laurent power series $\mathbf{C}((t))$: $\widehat{sl}_2 = sl_2 \otimes \mathbf{C}((t)) \otimes d$. For $\widehat{sl}_2$ we have Cartan decomposition $\widehat{sl}_2 = \mathbf{n}_+ \oplus \mathbf{h} \oplus \mathbf{n}_-$, where $\mathbf{n}_+$ is the nilpotent subalgebra $\mathbf{n}_+ = \mathbf{C}e \otimes 1 \oplus sl_2 \otimes \mathbf{C}[[t]]$, $\mathbf{n}_-$ is the opposite nilpotent subalgebra $\mathbf{n}_- = \mathbf{C}f \otimes 1 \oplus sl_2 \otimes \mathbf{C}[t^{-1}]$. Let G be the Lie group of the affine algebra $\widehat{sl}_2$, $N_\pm$ be the Lie group of the nilpotent subalgebra $\mathbf{n}_\pm$, $B_\pm$ be the Lie group of the Borel subalgebra $\mathbf{b}_\pm = \mathbf{n}_\pm \oplus \mathbf{h}$.

The group N_+ is isomorphic to the big cell X of the flag manifold $F = B_- \backslash G$, which is the orbit of 1 under the action of N_+. See references [8-10]. The Lie algebra $\widehat{sl}_2$ acts infinitesimally from the right by vector fields on F and hence on N_+. So does the Lie algebra $vect_- = \mathbf{C}[t^{-1}]t\partial_t$, with generators $L_n = t^{-n+1}\partial_t,\ n \geq 0$. Denote by ν the Lie algebra of vector fields on N_+. It contains two commuting Lie subalgebras: $\mathbf{n}^R_+$ and $\mathbf{n}^L_+$ of vector fields of the right and the left infinitesimal action of $\mathbf{n}_+$ on its Lie group. The vector field of the left infinitesimal action of $\beta \in \mathbf{n}_+$ on N_+ denoted by β^L. The Lie algebra $\mathbf{n}^R_+$ is a part of a larger subalgebra of ν, which is isomorphic to $\tilde{\mathbf{g}} = \widehat{sl}_2 \times vect_-$. The vector field of the right infinitesimal action of $\alpha \in \tilde{\mathbf{g}}$ on N_+ will be denoted by α^R.

For $j \in \mathbf{C}$ let M_j be Verma module over $\widehat{sl}_2$ of the sl_2 spin j with the highest vector v_j

$$\mathbf{n}_+ \cdot v_j = 0, \quad h \cdot v_j = jv_j, \quad M_j = U(\mathbf{n}_-) \cdot v_j.$$

M^*_j is the module contragradient to M_j with pairing $<,>$: $M^*_j \times M_j \to \mathbf{C}$. Following [2] we describe a geometrical construction of M^*_j. Let ω be Cartan anti-involution on $\widehat{sl}_2$, mapping $e, f \otimes t$ to $f, e \otimes t^{-1}$.

Define the right action of $y \in \widehat{sl}_2$ on $x \in M^*_j$ as follows:

$$< x \cdot y, z > = < x, \omega(y) \cdot z >, \quad z \in M_j$$

The module M_j^* can be identified with space of functions $\mathbf{C}(X)$ on the big cell X with respect to a twisted action. The right action of $\beta \in \widehat{sl}_2$ on M_j^* gives under this identification the action of

$$\beta^R + jF(\beta) \text{ on } \mathbf{C}(X),$$

where $F(\beta)$ is function on X. We have $F(h) = 1$ and $F(\beta) = 0$ for $\beta \in \mathbf{n}_+$.

Let vector v_m be singular vector of Verma module M_j , $v_m = P \cdot v_j$ for some element $P \in U(\mathbf{n}_-)$ and $R \cdot v_m = 0$ for any element $R \in U(\mathbf{n}_+)$. This singular vector defines homomorphism of $\widehat{sl}_2$-modules

$$i_P : M_m \to M_j, \quad u \cdot v_m \to (uP) \cdot v_j \tag{17}$$

for any $u \in U(\mathbf{n}_-)$. The map i_P commutes with the $\widehat{sl}_2$-action and is called intertwining opertor.

The left action of the element $\beta \in \mathbf{n}_+$ on $x \in M_j^*$ can be defined as follows

$$< \beta \cdot x, u \cdot v_j >=< x, (u\omega(\beta)) \cdot v_j >, \; u \in U(\mathbf{n}_-)$$

Let $\bar{P}$ be the image of $P \in U(\mathbf{n}_-)$ under isomorphism $U(\mathbf{n}_-) \to U(\mathbf{n}_+)$, which maps generators $e, f \otimes t$ to $f, e \otimes t^{-1}$. The homomorphism $\mathbf{n}_+ \to \nu$, mapping $\alpha \in \mathbf{n}_+$ to α^L , can be extended uniquly to homomorphism from $U(\mathbf{n}_+)$ to the algebra of differential operators on X . Let u^L be image of $u \in U(\mathbf{n}_+)$.

It is known [2] that homomorphism $i_P^* : M_j^* \to M_m^*$ dual to (17) can be realised as differential operator $\bar{P}^L$ on X. For example, consider the map i_f : $M_{-2} \to M_0$, $u \cdot v_{-2} \to (uf) \cdot v_0$. Then dual map $i_f^* : M_0^* \to M_{-2}^*$ is given by the left infinitesimal action e^L on $\mathbf{C}(X)$ treated as M_0^*.

Let us study further the left action of $U(\mathbf{n}_-)$ on $\mathbf{C}(X)$. In order to simplify formulas introduce the Chevalle basis of $\mathbf{n}_+ : e_1 = e, e_0 = f \otimes t$. For $F(\beta)$, $\beta \in \widehat{sl}_2$ we have following

$$[e_i^L, \beta^R] = 2(-)^i F(\beta) e_i^L, \quad i = 0, 1. \tag{18}$$

For proving (18) it's sufficient to consider e_1^L as $\widehat{sl}_2$-homomorphism from M_0^* to M_{-2} and e_0^L as $\widehat{sl}_2$-homomorphism from M_0^* to M_2 . Here we treat M_j^* as $\mathbf{C}(X)$ with twisted action. After identifing $\mathbf{C}(N_+)$ with $(U(\mathbf{n}_+))^*$ introduce grading on $\mathbf{C}(X) \simeq \mathbf{C}(N_+)$ with respect to degree of t and action of $\mathbf{h}$

$$deg\, e \otimes t^n = (n, 1), \; deg\, f \otimes t^n = (n, -1), \; deg\, h \otimes t^n = (n, 0) \tag{19}$$

We prove that the space of functions $\mathbf{C}(N_+/H_+)$ on the homogeneous space N_+/H_+ is isomorphic to π_0 as $\mathbf{n}_+$-modules. Indeed, vector fields Q_1, Q_0 (15) define structure of $\mathbf{n}_+$-module on π_0. Let x_i, y_i be coordinates on π_0

$$x_{i+1} = \partial^i \beta, \quad y_i = \partial^i \gamma, \quad i \geq 0$$

Then for the vector fields $Q_1,\ Q_0$ on π_0 we have (see (15))

$$
\begin{aligned}
Q_1 &= \frac{\partial}{\partial y_0} + x_1 y_0 \frac{\partial}{\partial y_1} + (x_1^2 y_0^2 + x_2 y_0 + x_1 y_1)\frac{\partial}{\partial y_2} + \cdots \\
Q_0 &= \frac{\partial}{\partial x_1} - x_1 y_0 \frac{\partial}{\partial x_2} + (x_1^2 y_0^2 - x_2 y_0 + x_1 y_1)\frac{\partial}{\partial x_3} + \cdots
\end{aligned}
\tag{20}
$$

Let X be an operator on π_0 of the form $\sum_i (X_i \partial/\partial x_i + Y_i \partial/\partial y_i)$, then define its shift term as all the terms $X_i \partial/\partial x_i$ or $Y_i \partial/\partial y_i$, for which X_i or Y_i is constant. From (20) shift terms of Q_1 and Q_0 are $\partial/\partial y_0$ and $\partial/\partial x_1$. The shift term of $[Q_1, Q_0]$ is 0. Q_1 and Q_0 satisfy Serre relation and generate $\mathbf{n}_+$. In our notation we can treat Q_1 as e and Q_0 as $f \otimes t$. It's easy to see that vector fields corresponding to subalgebra $\mathbf{h}_+$ have no shift terms, and shift term of $e \otimes t^n$ is $\partial/\partial y_n, \quad n \geq 0$, while shift term of $f \otimes t^n$ is $\partial/\partial x_n, \quad n \geq 1$.

Consider the module π_0^* over $\mathbf{n}_+$, dual to π_0. We can identify π_0 with π_0^* as linear spaces, choosing the monomials $x_{k_1}/k_1! \cdots x_{k_n}/k_n!$ and $y_{k_1}/k_1! \cdots y_{k_n}/k_n!$ as an ortonormal basis. The formulas for the action of $\mathbf{n}_+$ on π_0^* are obtained from the formulas for its action on π_0 by interchainging x_n (y_n) and $\partial/\partial x_n$ (resp.$\partial/\partial y_n$). Combinations of $Q_0,\ Q_1$, corresponding to $\mathbf{h}_+$, act on $1^* \in \pi_0^*$ by 0 (because they have not shift terms).

Let us introduce

$$
N = U(\mathbf{n}_+) \otimes_{U(\mathbf{h}_+)} \mathbf{C},
$$

$\mathbf{n}_+$-module, induced from the trivial one-dimentional representation of subalgebra $\mathbf{h}_+$. Since action of $\mathbf{h}_+$ on $1 \otimes 1 \in N$ is trivial, there is unique $\mathbf{n}_+$-homomorphism $N \to \pi_0$, sending $1 \otimes 1 \in N$ to $1^* \in \pi_0^*$, and $(e \otimes t^{n_1}) \cdots (e \otimes t^{n_k})(f \otimes t^{m_1}) \cdots (f \otimes t^{m_l}) \otimes 1$ maps to $y_{n_1} \cdots y_{n_k} x_{m_1} \cdots x_{m_k} \cdot 1^*$ + lower order terms. Therefore map $N \to \pi_0$ has no kernel. N and π_0 coinside as linear spaces with respect to graiding (16) and (19). Thus

$$
\pi_0 \simeq \left(U(\mathbf{n}_+) \otimes_{U(\mathbf{h}_+)} \mathbf{C}\right)^* \simeq \mathbf{C}(N_+/H_+)
\tag{21}
$$

as $\mathbf{n}_+$ -modules.

Recall that $\mathbf{h}_\pm$ is commuting subalgebra of $\mathbf{n}_\pm$, generated by $h_{\pm i} = h \otimes t^{\pm i}, \quad i = 1, 2, \cdots$. Consider eq. (18) for $\beta = h_{-1}$

$$
[e_i^L, h_{-1}^R] = (-)^i \, 2F(h_{-1}) e_i^L, \quad i = 0, 1
\tag{22}
$$

as vector fields on $\mathbf{C}(N_+)$. For the function $F(h_{-1})$ on the group N_+ we have

$$
x_+^R \cdot F(h_{-1}) = 0,
\tag{23}
$$

for $x_+ \in \mathbf{h}_+$. This can be derived from commutator (22). Indeed,

$$
[x_+^R, [e_i^L, h_{-1}]] = -[e_i^L, [h_1^R, x_+^R]] - [h_{-1}^R, [x_+^R, e_i^L]] = 0.
$$

So for $i=0,1$ $(x_+^R \cdot F(h_{-1})) \cdot e_i^L = 0$ and then (23). Eq. (23) gives us example of $\mathbf{h}_+^R$- invariant functions on $\mathbf{C}(X)$. It's obvious that the right action of $\mathbf{h}_-$ on $F(h_{-1})$ preserves $\mathbf{h}_+$-invariancy, thus $\beta^R \cdot F(h_{-1})$ for $\beta \in \mathbf{h}_-$ is $\mathbf{h}_+$ -invariant. Other families of $\mathbf{h}_+$ invariant functions on $\mathbf{C}(X)$ are given by $F(f)$, $F(e \otimes t^{-1})$ and the right action of $\mathbf{h}_-$ on them.

Thus we can treat (22) as vector fields on $\mathbf{C}(N_+/H_+)$, because e_i^L, h_{-1}^R commutes with $\mathbf{h}_+^R$, and $F(h_{-1})$ is $\mathbf{h}_+$-invariant.

Using isomorphism (21) we can identify e_i^L acting on $\mathbf{C}(N_+/H_+)$ with Q_i acting on π_0. The element $F(h_{-1})$ of $\mathbf{C}(N_+/H_+)$ has grading (16): $degF(h_{-1}) = (-1;0)$. The only image of $F(h_{-1})$ in π_0 under isomorphism $\mathbf{C}(N_+/H_+) \to \pi_0$ of the same degree is $const \cdot x_1 y_0 (= const \cdot \beta\gamma)$, $deg x_1 y_0 = (-1;0)$.

So using (22) and isomorphism (21), we have

$$[Q_i, \eta_{-1}] = const \cdot (-)^i \, 2 x_1 y_0 Q_i, \quad i = 0,1 \tag{24}$$

as vector fields on π_0, where η_{-1} is unknown vector field on π_0.

We prove that the vector field η_{-1} is proportional to the vector field of derivative $\partial = \sum x_i \partial/\partial x_{i-1} + y_i \partial/\partial y_{i-1}$. If we choose $const = -\frac{1}{2}$ then

$$\eta_{-1} = \partial \tag{25}$$

on π_0. Indeed, operators $\bar{Q}_1 = TQ_1$ and $\bar{Q}_0 = T^{-1}Q_0$ commute the action of ∂ (11) on π_0. That is why

$$[Q_1, \partial] = T^{-1} [\partial, T] \, Q_1 = x_0 y_1 Q_1 \;\text{ and }\; [Q_0, \partial] = T \, [\partial, T^{-1}] = -x_0 y_1 Q_0$$

So the image of the action of h_{-1}^R on $\mathbf{C}(N_+/H_+)$ under isomorphism (21) is the action of ∂ on π_0.

4. Cohomology Computation

In this section we describe the space of IM of MB eq. using a double complex [2]. Let $B^*(\widehat{sl}_2)$ be the dual of BGG resolution for $\widehat{sl}_2$ [11]

$$B^*(\widehat{sl}_2) = \bigoplus_{j \geq O} B^j(\widehat{sl}_2),$$

where $B^0(\widehat{sl}_2) = M_0^*$ and $B^j(\widehat{sl}_2) = M_{2j}^* \oplus M_{-2j}^*$. Here M_j^* is contragradient module to the Verma one with sl_2 spin j and level k=0.

The Verma module M_0 contains singular vectors, labeled by the elements of the Weyl group of $\widehat{sl}_2$. Denote this vectors by w_0 and $w_{\pm j}$, $j = 1, 2, \cdots$. The weights of $w_{\pm j}$ are $\pm 2j$. The action of $U(\mathbf{n}_-)$ on $w_{\pm j}$ generates the submodule $M_{w_{\pm j}}$ isomorphic to $M_{\pm 2j}$. Let $l(w_{\pm j}) = j$. Choose two elements w , w' of the Weyl group, such that $l(w) = l(w') + 1$. Then $\bar{P}^L_{w,w'}$ is a map, dual to embeding (17) $i_{w,w'}$: $M_w \to M_{w'}$. (We identify M_j^* with $\mathbf{C}(X)$).

The 0-th cohomology of $B^*(\widehat{sl}_2)$ is one-dimentional and all higher ones vanish. Differentials $\delta^j : B^j(\widehat{sl}_2) \to B^{j+1}(\widehat{sl}_2)$ of $B^*(\widehat{sl}_2)$ can be written in a common way:

$$\delta^j = \sum_{l(w)=j, l(w')=j+1} \epsilon_{w,w'} \bar{P}^L_{w,w'}.$$

From this formula the first differential δ^1, mapping $M_0^* \to M_2^* \oplus M_{-2}^*$ is $e_1^L - e_0^L$ as the action on $\mathbf{C}(X)$.

The right action of $\widehat{sl}_2$ on complex $B^*(\widehat{sl}_2)$ commutes with the differentials δ^j. This valid for $\mathbf{h}_+ \subset \widehat{sl}_2$. So we can take quotient of $B^*(\widehat{sl}_2)$ by the right action of Lie subgroup H_+ of N_+. Denote this complex by $F^*(\widehat{sl}_2)$:

$$F^*(\widehat{sl}_2) = \bigoplus_{j \geq 0} F^j(\widehat{sl}_2),$$

where $F^0(\widehat{sl}_2) = \pi^{(0)}$ and $F^j(\widehat{sl}_2) = \pi^{(2j)} \oplus \pi^{(-2j)}$, here $\pi^{(\pm 2j)}$ denotes space of $\mathbf{h}_+$-invariants of $M^*_{\pm 2j}$.

The right action of $\mathbf{h}_-$ on $B^*(\widehat{sl}_2)$ gives rise to $\mathbf{h}_-$-action on $F^*(\widehat{sl}_2)$, because for $x \in \mathbf{h}_-$ function $F(x)$ is $\mathbf{h}_+$-invariant . The action of $x \in \mathbf{h}_-$ on $\pi^{(j)}$ is given by

$$x^R + jF(x) \tag{26}$$

This action commutes with the differentials of the complex $F^*(\widehat{sl}_2)$, that is why the action of $\mathbf{h}_-$ is defined on the cohomologies of $F^*(\widehat{sl}_2)$.

For description of the space of IM of MB eq. cohomologies of $F^*(\widehat{sl}_2)$ should be computed. We prove that cohomologies of $F^*(\widehat{sl}_2)$ are isomorphic to $\wedge^*(\mathbf{h}_+^*)$.

Since $B^*(\widehat{sl}_2)$ is injective resolution of the trivial representation of $\mathbf{n}_+$, the cohomologies of $F^*(\widehat{sl}_2)$ coincide with the cohomologies $H^*(\mathbf{n}_+, \pi_0)$ of the Lie algebra $\mathbf{n}_+$ with coefficients in the module π_0 [1]. Because π_0 can be identified with $\left(U(\mathbf{n}_+) \otimes_{U(\mathbf{h}_+)} \mathbf{C}\right)^*$, we have by Shapiro lemma:

$$H^*(\mathbf{n}_+, \pi_0) \simeq H^*(\mathbf{h}_+, \mathbf{C}) \simeq \wedge^*(\mathbf{h}_+^*) \tag{27}$$

In [2] it was proved that the action of $\mathbf{h}_-$ on $H^*(\mathbf{n}_+, \pi_0)$ is trivial. In particular, the operator h_{-1} acts trivially on the cohomologies. We already know that its action on π_0 consides with the action of ∂. Consider now the action of h_{-1} on $\pi^{(\pm 2j)}$. It is given by $h_{-1}^R \pm 2jF(h_{-1})$. The image of $F(h_{-1})$ under isomorphism $\mathbf{C}(N_+/H_+) \to \pi_0$ is equal to $-\frac{1}{2}\beta\gamma$. So the action of h_{-1} on $\pi^{(\pm 2j)}$ coincides with the action (11) of derivative ∂ on $\pi_{\mp j}$ and we have isomorphism $\pi^{(\pm 2j)} = \pi_{\mp j}$ with respect to the action of $\mathbf{n}_+$. Thus $F^0(\widehat{sl}_2) = \pi_0$, $F^j(\widehat{sl}_2) = \pi_{-j} \oplus \pi_j$ and the first differential δ^1: $\pi_0 \to \pi_{-1} \oplus \pi_1$ is equal to $\delta^1 = \bar{Q}_1 - \bar{Q}_0$. We see that differential δ^1 coincides with the action of MB hamiltinian on π_0. This observation enables

us to compute the space of IM of MB eq. Indeed, the space of IM is isomorphic to the kernel of the map $\delta^1 = \bar{Q}_1^* - \bar{Q}_0^*$: $\hat{\pi}_0 \to \hat{\pi}_{-1} \oplus \hat{\pi}_1$. Recall that $\hat{\pi}_n = \pi_n / \partial \pi_n$. In order to use the complex $F^*(\widehat{sl}_2)$ for computing IM we should get rid of total derivatives in z, i.e. consider $F^*(\widehat{sl}_2)/\partial F^*(\widehat{sl}_2)$. For these reasons a double complex is used.

Now the main result can be formulated : the space of integrals of motion of the MB eq. (8) is lineary spanned by elements H_m, $m = 1, 2, \cdots$, where $deg H_m = (-m, 0)$.

Since $\partial = h_{-1}$ commutes with the differentials of $F^*(\widehat{sl}_2)$, we can consider the doube complex:

$$\mathbf{C} \longrightarrow F^*(\widehat{sl}_2) \stackrel{\pm h_{-1}}{\longrightarrow} F^*(\widehat{sl}_2) \longrightarrow \mathbf{C} \tag{28}$$

Using the spectral sequence, in which h_{-1} is the 0 th differential , one gets that the 1st cohomology of the double complex H^1_{tot} is isomorphic to the space of IM. We can also compute this cohomology using the other spectral sequence. Because h_{-1} acts trivially on $H^1(F^*(\widehat{sl}_2))$, $H^1_{tot} \simeq H^1(F^*(\widehat{sl}_2)) \simeq \mathbf{h}_+^*$. Therefore space of IM is linearly spanned by elements H_m of degree (s=-m , q=0) for m=1,2$\cdots$.

6. Construction of IM and vector fields

Now we show explicitly how IM are connected with the first cohomology class $H^1(F^*(\widehat{sl}_2))$. Consider the element $\mathcal{H}$ of cohomology class $H^1(F^*(\widehat{sl}_2)) \subset \pi_{-1} \oplus \pi_1$. If we apply ∂ to $\mathcal{H}$, we obtain a trivial cycle. So there exists element $h \in \pi_0$ such that

$$\delta^1 \cdot h = \partial \mathcal{H}, \tag{29}$$

where δ^1 is the first differential of the complex $F^*(\widehat{sl}_2)$ equal to the MB flow: $\delta^1 = \bar{Q}_1 - \bar{Q}_0 = \partial_\tau$. By construction h is not total derivative, so it belongs to IM. Thus, formula (29) explicitly gives isomorphism between the spase of IM and the cohomology class $H^1(F^*(\widehat{sl}_2))$.

For example, consider two lowest IM: $h^{(-2)} = rq \in \pi_0$ and $h^{(-3)} = r\partial q \in \pi_0$ of degrees (-2; 0) and (-3; 0). They are connected via formula (29) with $j^{(-1)} = \frac{1}{4}\gamma e^{-\int^z dt\, \beta(t)\gamma(t)} \in H^1(F^*(\widehat{sl}_2))$ and $j^{(-2)} = \frac{1}{4}\beta\gamma^2 e^{-\int^z dt\, \beta(t)\gamma(t)} \in H^1(F^*(\widehat{sl}_2))$ of degrees (-1; 0) and (-2; 0).

Denote by $H_m \in \hat{\pi}_0$ element of the space of IM of MB eq. of degree (-m; 0). Let η_m be the vector field on π_0:

$$\eta_m = \{\cdots, H_m\}^* . \tag{30}$$

It is easy to see that $\eta_1 = \partial$. We can treat η_m as vector fields on N_+/H_+ under isomorphism $\pi_0 \simeq \mathbf{C}(N_+/H_+)$. On the other hand the right action of generator $h_{-m}, \quad m = 1, 2, \cdots$ of the subalgebra $\mathbf{h}_- \in \widehat{sl}_2$ on N_+/H_+ also defines vector

fields μ_m on N_+/H_+. Now we formulate the result : the vector fields μ_m coincide with η_m, $m = 1, 2 \cdots$.

The statement was already proved for m=1. We have shown that action of h_{-1} on π_0 gives us ∂. So does vector field $\eta_1 = \{\cdots, H_1\}^*$, where $H_1 = \oint dzrq$. Consider now commutator of operators $\bar{Q}_1$ and η_m, acting on $P \in \pi_0$

$$[\bar{Q}_1, \eta_m]P = \left\{ \{ \oint dz \beta e^{\int^z dt\, \beta(t)\gamma(t)}, H_m\}^*, P \right\}^* = 0,$$

since $\{\oint dz \beta e^{\int^z dt\, \beta(t)\gamma(t)}, H_m\}^* = 0$ by the definition of IM. The same is valid for $\bar{Q}_0$.

Now we can compute commutator of vector fields Q_i and η_m on π_0. Using expressions for $\bar{Q}_1 = TQ_1$ and $\bar{Q}_0 = T^{-1}Q_0$ one can derive:

$$[Q_i, \eta_m] = -f^i_m Q_i, \tag{31}$$

as vector fields on π_0, where f^i_m —certain function on π_0.

Let us consider now eq.(18) on $\mathbf{C}(N_+)$ and vector fields β on $\mathbf{C}(N_+)$, satisfing it. It is obvious that, if vector fields α and β satisfy (18), then $[\alpha, \beta]$ satisfies it too. So we have Lie algebra of vector fields on $\mathbf{C}(N_+)$, satisfing (18). In [2] it was shown, that this Lie algebra is isomorphic to $\tilde{\mathbf{g}}$. The only vector fields, satisfing (18), which in addition $\mathbf{h}_+$-invariant, are generated by right action of $\mathbf{h}_-$ on $\mathbf{C}(N_+)$, i.e. μ_m. On the other hand vector fields η_m can be lifted to $\mathbf{h}_+$-invariant vector fields $\tilde{\eta}_m$ on N_+, which satisfy relation (31) and thus (18). Comparing the degrees of vector fields $\tilde{\eta}_m$ and μ_m we get the statement.

Now proof of well- known result [7]: flows generated by IM of MB eq. (i.e. the positive AKNS hamiltonians) commutes with each other $\{H_m, H_n\} = 0$ for $m, n = 1, 2, \cdots$ is obvious.

Because of commutativity of subalgebra $\mathbf{h}_-$ the vector fields μ_m , generating by the right action of $h_{-m} \in \mathbf{h}_-$ commute with each other. So do corresponding vector fields η_m on π_0 . Using definition of η_m (30) , we find commutativity of the positive AKNS hamiltonians.

Concluding Remarks

In this paper we showed the relation between classical Maxwell-Bloch equation and AKNS hierarchy with the geometry of the affine Lie group coset N_+/H_+. The simplisity of the action of the Maxwell-Bloch hamiltonian and the integrals of motion on the phase space enables to treate coordinates on the coset N_+/H_+ as "scatering data". The technique of the Lie group interpretation of the phase space was transmited from classical Toda theories [1-2] to the classical Maxwell-Bloch equation. It is possible to give natural Poisson group structure on the Maxwell-Bloch phase space contrary to the Toda phase space. Namely, the action of the hamiltonian on the MB phase space can be understood as a Poisson action of a

Poisson-Lie group on a Poisson manifold. Quantization of such Poisson structures leads to a quantum version of Maxwell-Bloch equation. Another way to quantize Maxwell-Bloch equation is to use vertex operator algebra methods as it was done for Toda theories [1].

The lattice version of MB eq. can be given as follows. Consider functions

$$X(z) = \beta(z)e^{\phi(z)} \text{ and } Y(z) = \gamma(z)e^{-\phi(z)}$$

and introduce $X_i = X(ia)\,, Y_i = Y(ia), \quad i \in \mathbf{Z}$ as lattice variables, where a is a lattice parameter. The Poisson bracket of X_i and Y_j computed using definition (1) is a classical limit of a q- commutator of lattice variables $[X_i\,,\ Y_j]_q = \delta_{ij}$. We will describe the lattice version of MB eq. in the forthcoming paper.

Such kind of latticization was proposed by one of the authors (B. F) [12] and studied for Toda systems in [13,14].

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